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# The massive scalar meson field in a Schwarzschild background space

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**Abstract.** The radial equation of a static massive scalar meson field in a Schwarzschild background space is solved by asymptotic methods leading to solutions over the whole range. This approach is used to obtain the form of the Yukawa potential in the presence of a large Schwarzschild black hole, and to show that the meson field fades away to zero as the event horizon is reached, in agreement with the 'no hair' conjecture.

## 1. Introduction

The electrostatic field of a point charge in a Schwarzschild space has been fully discussed by Cohen and Wald (1971) and Hanni and Ruffini (1973). This work shows that the slow fall of the charge into a Schwarzschild black hole leads, in the limit as the event horizon is reached, to the formation of a Nordström–Reissner black hole. The corresponding problem of the slow fall of a strongly interacting nuclear source (a baryon) has been discussed by Teitelboim (1972), who found that the associated massive scalar meson field fades away to zero as the event horizon is reached. This result is in agreement with the well known 'no hair' conjecture that a black hole can only possess mass, electric charge and angular momentum. Teitelboim's work, however, did not follow the method used by Cohen and Wald and Hanni and Ruffini, and depended only on the general nature of the solution rather than the full solution of the scalar meson field equation in a Schwarzschild space. Accordingly no explicit form was obtained for the meson field of a baryon in the curved space.

In this paper a detailed solution of the Klein–Gordon equation in a Schwarzschild space is found, and the form of the Yukawa potential due to a static source is obtained. However, unlike the electrostatic problem, the basic radial equation is not soluble in terms of known functions, and furthermore the existence of two constants representing the masses of the meson and the black hole leads to the appearance of a large dimensionless constant in the basic differential equation. This equation is not soluble exactly but is amenable to the method of asymptotics developed by Green and Liouville (see Jeffreys 1962 and Olver 1974).

## 2. Basic equations

In flat space the static meson field due to a baryon source at $\boldsymbol{x}=\boldsymbol{x}'$ is described by the Klein–Gordon equation in its time-independent form

$$(\nabla^2-\mu^2)\Phi=g\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'), \tag{2.1}$$

where $g$ is the source strength, and $\mu$ is the inverse Compton wavelength of the $\pi$-meson. This equation has the well known (Yukawa) spherically symmetric potential solution $\Phi = -g\, e^{-\mu r}/r$ outside the source. Equation (2.1) may be put into the generally covariant form

$$(\Box^2 + \mu^2)\Phi = -g \int_{-\infty}^{\infty} \frac{\delta^{(4)}(x - x'(\lambda))}{\sqrt{-g_4}}\, ds(\lambda) \tag{2.2}$$

(see Teitelboim 1972), where $x'(\lambda)$ defines the world line of the source such that $x^0 = x'^0(\lambda)$ defines $\lambda$ in terms of $x^0$, the time coordinate. Throughout we use the metric $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ $(\mu, \nu = 0, 1, 2, 3)$ with a signature $-2$, and write $g_4$ as the determinant of $g_{\mu\nu}$. If the source is stationary then $ds(\lambda) = \sqrt{g_{00}}\, dx'^0$ and the right-hand side of (2.2) becomes

$$-g \int_{-\infty}^{\infty} \frac{\delta^{(4)}(x - x')}{\sqrt{-g_4}} \sqrt{g_{00}}\, dx'^0 = -g \sqrt{\frac{g_{00}}{-g_4}}\, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'). \tag{2.3}$$

If the space and time parts of the metric are orthogonal then (2.3) reduces to

$$-g \frac{\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')}{\sqrt{-g_3}} \tag{2.4}$$

where $g_3$ is the determinant of the metric tensor of the space part.

We now consider a baryon source outside a black hole situated at $r = b$, $\theta = 0$ in a background space given by the Schwarzschild exterior metric

$$ds^2 = [1 - (2m/r)]\, dt^2 - [1 - (2m/r)]^{-1}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2, \tag{2.5}$$

where $m$ is the mass of the collapsed object in appropriate units. Taking $\Phi$ to be independent of time $t$ $(= x^0)$ and $\phi$, (2.2) and (2.4) lead to the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(1 - \frac{2m}{r}\right)\frac{\partial \Phi}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) - \mu^2 \Phi$$
$$= g\frac{\delta(r - b)\, \delta(\cos\theta - 1)}{2\pi r^2}\left(1 - \frac{2m}{r}\right)^{1/2}, \tag{2.6}$$

where $\int \delta(r - b)\, dr = 1$ and $\int \delta(\cos\theta - 1) \sin\theta\, d\theta\, d\phi = 2\pi$. Writing

$$\Phi = \sum_{l=0}^{\infty} R_l(r) P_l(\cos\theta) \tag{2.7}$$

and substituting into (2.6) we find

$$\sum_{l=0}^{\infty} P_l(\cos\theta)\left\{\frac{d}{dr}\left[r^2\left(1 - \frac{2m}{r}\right)\frac{dR_l}{dr}\right] - [l(l+1) + \mu^2 r^2] R_l\right\}$$
$$= g\frac{\delta(r - b)\, \delta(\cos\theta - 1)}{2\pi}\left(1 - \frac{2m}{r}\right)^{1/2}, \tag{2.8}$$

which, on multiplying by $P_{l'}(\cos\theta) \sin\theta\, d\theta$ and integrating over $\theta$, leads to the radial equation

$$\frac{d}{dr}\left(r(r - 2m)\frac{dR_l}{dr}\right) - [l(l+1) + \mu^2 r^2] R_l = g\frac{(2l+1)}{4\pi}\left(1 - \frac{2m}{r}\right)^{1/2} \delta(r - b). \tag{2.9}$$

The substitution $x=(r-2m)/m$ into (2.9) finally gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(x+2)\frac{\mathrm{d}R_l}{\mathrm{d}x}\right)-[l(l+1)+N^2(x+2)^2]R_l=g\frac{(2l+1)}{4\pi m}\left(\frac{x}{x+2}\right)^{1/2}\delta(x-x_b), \qquad (2.10)$$

where $N=\mu m$, and $x_b=(b-2m)/m$. The appearance of the constant $N$ arises from the non-zero rest mass of the quantum of the meson field. In the work on the electrostatic field no such constant appeared since the quantum of the electromagnetic field (the photon) has zero rest mass. We note here that for an object of Sun-like mass with a Schwarzschild radius $2m$ of order $10^5$ cm, $N$ is of order $10^{18}$ since $\mu \simeq 10^{13}\ \mathrm{cm}^{-1}$.

## 3. Solution of the radial equation outside the source

When $x \neq x_b$, (2.10) may be written in normal form by putting

$$R_l=Z_l[x(x+2)]^{-1/2}$$

to give

$$\frac{\mathrm{d}^2Z_l}{\mathrm{d}x^2}-\left[N^2\left(\frac{2+x}{x}\right)+\frac{l(l+1)}{x(x+2)}-\frac{1}{x^2(x+2)^2}\right]Z_l=0. \qquad (3.1)$$

This equation does not appear to be exactly soluble in terms of known functions, and accordingly we look for an approximate solution by using the Green–Liouville method of asymptotics as follows: the independent variable $x$ is changed to $\xi$ by the transformation $x=x(\xi)$, and the dependent variable changed to $G_l=(\mathrm{d}\xi/\mathrm{d}x)^{1/2}Z_l$. In this way we obtain the equation

$$\frac{\mathrm{d}^2G_l}{\mathrm{d}\xi^2}=\left\{\left[N^2\left(\frac{2+x}{x}\right)+\frac{l(l+1)}{x(x+2)}-\frac{1}{x^2(x+2)^2}\right]\frac{1}{\xi'^2}+\frac{\xi'''}{2\xi'^3}-\frac{3}{4}\frac{\xi''^2}{\xi'^4}\right\}G_l. \qquad (3.2)$$

Letting $k^2=N^2+l(l+1)$, $\beta^2=N^2/k^2$ and $\alpha^2=l(l+1)/k^2$, so that $0\leqslant\alpha\leqslant1$, $0\leqslant\beta\leqslant1$, and $\alpha^2=1-\beta^2$, (3.2) becomes

$$\frac{\mathrm{d}^2G_l}{\mathrm{d}\xi^2}=\left[\left\{k^2\left[\beta^2\left(\frac{2+x}{x}\right)+\frac{\alpha^2}{x(x+2)}\right]-\frac{1}{x^2(x+2)^2}\right\}\frac{1}{\xi'^2}+\frac{\xi'''}{2\xi'^3}-\frac{3}{4}\frac{\xi''^2}{\xi'^4}\right]G_l, \qquad (3.3)$$

where the relation between $x$ and $\xi$ is at our disposal. Choosing

$$\xi'^2=\beta^2\left(\frac{2+x}{x}\right)+\frac{\alpha^2}{x(x+2)} \qquad (3.4)$$

we have

$$\xi=\int_0^x\left[\beta^2\left(\frac{2+x}{x}\right)+\frac{\alpha^2}{x(x+2)}\right]^{1/2}\mathrm{d}x, \qquad (3.5)$$

which may be expressed as the sum of a number of elliptic integrals. Equation (3.4) allows (3.3) to be written as

$$\frac{\mathrm{d}^2G_l}{\mathrm{d}\xi^2}=\left(k^2-\frac{1}{4\xi^2}+g_1(\xi)\right)G_l, \qquad (3.6)$$

where

$$g_1(\xi)=\left[\frac{\xi'''}{2\xi'^3}-\frac{3}{4}\frac{\xi''^2}{\xi'^4}-\frac{1}{x^2(x+2)^2}\frac{1}{\xi'^2}\right]+\frac{1}{4\xi^2} \tag{3.7}$$

$$=\frac{\beta^4(x+2)^4(4x-1)+2\alpha^2\beta^2(x+2)^2(3x^2+3x-1)+\alpha^2(x^2+2x-1)}{4[\beta^2(x+2)^2+\alpha^2]^3x(x+2)}+\frac{1}{4\xi^2}. \tag{3.8}$$

After some lengthy calculations, we find that $g_1(\xi)=O(1)$ as $\xi\to 0$ and is $O(1/\xi^2)$ as $\xi\to\infty$. Furthermore, numerical calculations show that $g_1(\xi)$ has an upper bound of less than 5 and is a slowly varying function of $\xi$. Now since $k^2=N^2+l(l+1)$ is a large parameter we may obtain very good approximate solutions of (3.6) by neglecting $g_1(\xi)$. The resulting equation has two solutions

$$G_l(\xi)=\begin{cases}\xi^{1/2}I_0(k\xi),\\ \xi^{1/2}K_0(k\xi),\end{cases} \tag{3.9}$$

where $I_0$ and $K_0$ are the modified Bessel functions of the first and second kind respectively. Using the two transformations relating $R_l$ to $Z_l$, and $Z_l$ to $G_l$, and the expression for $\xi'$ in (3.4), we have

$$R_l(x)\simeq\frac{1}{[x(x+2)]^{1/2}}\frac{[x(x+2)]^{1/4}}{[\beta^2(x+2)^2+\alpha^2]^{1/4}}\xi^{1/2}\begin{cases}I_0(k\xi),\\ K_0(k\xi),\end{cases} \tag{3.10}$$

which defines the two solutions

$$R_l^{(1)}(x)=\frac{\xi^{1/2}I_0(k\xi)}{[x(x+2)]^{1/4}[\beta^2(x+2)^2+\alpha^2]^{1/4}} \tag{3.11}$$

and

$$R_l^{(2)}(x)=\frac{\xi^{1/2}K_0(k\xi)}{[x(x+2)]^{1/4}[\beta^2(x+2)^2+\alpha^2]^{1/4}}. \tag{3.12}$$

## 4. Integration across the singularity

We now use the approximate solutions (3.11), (3.12) to obtain a solution of (2.10). First integrate (2.10) across the singularity and take the limit as the range of integration goes to zero. The requirement of continuity on $R_l(x)$ at $x=x_b$ gives

$$\left.\frac{\mathrm{d}R_l}{\mathrm{d}x}\right|_{x_b+0}-\left.\frac{\mathrm{d}R_l}{\mathrm{d}x}\right|_{x_b-0}=g\frac{(2l+1)x_b}{4\pi m[x_b(x_b+2)]^{3/2}}. \tag{4.1}$$

The required solution of (2.10) should be finite as $x\to 0$ and approach zero as $x\to\infty$. This implies that $R_l(x)$ must have the form

$$R_l(x)=\begin{cases}AR_l^{(1)}(x), & 0\leqslant x\leqslant x_b\\ BR_l^{(2)}(x), & x_b\leqslant x<\infty.\end{cases} \tag{4.2}$$

Substituting (4.2) into (4.1) and using continuity at $x=x_b$ leads to

$$R_l(x)=-g\frac{(2l+1)}{4\pi m}\left(\frac{x_b}{x_b+2}\right)^{1/2}\begin{cases}R_l^{(2)}(x_b)R_l^{(1)}(x), & 0\leqslant x\leqslant x_b,\\ R_l^{(1)}(x_b)R_l^{(2)}(x), & x_b\leqslant x<\infty.\end{cases} \tag{4.3}$$

Finally inserting (4.3) into (2.7) we have

$$\Phi(x,\theta)=\begin{cases}\dfrac{-g}{4\pi m}\displaystyle\sum_{l=0}^{\infty}\left(\frac{x_b}{x_b+2}\right)^{1/2}(2l+1)R_l^{(2)}(x_b)R_l^{(1)}(x)\mathrm{P}_l(\cos\theta) & 0\leqslant x\leqslant x_b\\ \dfrac{-g}{4\pi m}\displaystyle\sum_{l=0}^{\infty}\left(\frac{x_b}{x_b+2}\right)^{1/2}(2l+1)R_l^{(1)}(x_b)R_l^{(2)}(x)\mathrm{P}_l(\cos\theta) & x_b\leqslant x<\infty\end{cases}\tag{4.4}$$

which when written out specifically for $x_b < x < \infty$ has the form

$$\Phi(x,\theta)=-\frac{g}{4\pi m}\sum_{l=0}^{\infty}\left(\frac{x_b}{x_b+2}\right)^{1/2}\times\frac{(2l+1)(k\xi)^{1/2}K_0(k\xi)(k\xi_b)^{1/2}I_0(k\xi_b)\mathrm{P}_l(\cos\theta)}{[x(x+2)]^{1/4}[N^2(x+2)^2+l(l+1)]^{1/4}[x_b(x_b+2)]^{1/4}[N^2(x_b+2)^2+l(l+1)]^{1/4}},\tag{4.5}$$

where $\xi_b$ is the value of $\xi$ at $x = x_b$, and $x = (r-2m)/m$. The uniform convergence of this series with respect to $x_b$ has been proved for $0 \leqslant x_b < x$, and $0 < x < \infty$. Since each term in the series tends to zero as $x_b \to 0$, we see that $\Phi(x, \theta)$ also tends to zero as $x_b \to 0$, that is, as the source approaches the event horizon. Further the dominant term in the fall off, for small $x_b$, is found to be given by

$$\left(\frac{x_b}{x_b+2}\right)^{1/2}=\left(1-\frac{2m}{b}\right)^{1/2}\qquad(\text{as } b\to 2m)\tag{4.6}$$

in agreement with the result found by Teitelboim. The explicit solutions obtained here may have some interest in the quantization programme. We should finally emphasize again that the analysis used here depends on the existence of a large parameter in the theory and is therefore probably not appropriate for the study of small black holes where $N$ will be a much smaller quantity.

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